AN UPDATED SURVEY ON SOME RESULTS ON ENERGY OF GRAPHS

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ABSTRACT

The eigenvalue of a graph G is the eigenvalue of its adjuency matrix. The energy E(G) of G is the sum of absolute values of the elgenvalues. A natural question arises: How the energy of a given graph G can be related with the graph obtained from G by means of some graph operations? In order to answer this question, we have considered two graphs namely, splitting graph $S(G)$ and shadow graph $D_2(G)$. It has been proven that $E\left(S'(G)\right) = \sqrt{5}E(G)$ and $E\left(D_2(G)\right) = 2E(G)$.

1. INTRODUCTION

All graphs considered here are simple, finite and undirected. For standard terminology and notations related to graph theory, we follow Balakrishnan and Ranganathan [2] while for algebra we follow Lang [7].

The Adjacency matrix A(G) of a graph G with vertices $v_1,\ v_2 \ldots\ldots v_n$ is an n × n matrix $[a_{ij}]$ such that $a_{ij} = 1$ if v_i is adjacent to v_j , and 0 otherwise.

The eigenvalues $\lambda_1, \lambda_2 \ldots\ldots \lambda_n$ of the graph G are the eigenvalues of its adjacency matrix. The set of eigenvalues of the graph with their multiplicities is known as spectrum of the graph.

In 1978 Gutman [5] defined the energy of a Graph G as the sun of absolute values of the eigenvalues of graph G and denoted it by E(G), Hence,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

In 2004, Bapat and Pati [3] proved that if the energy of a graph is rational then it must be an even integer, while Pirzada and Gutman [9] established that the energy of a graph is never the square root of an odd integer. A brief account of graph energy is given in [1] as well as in the books[4,8]. Some fundamental results on graph energy are also reported in the thesis of Sriraj [10].

The present work is intended to relate the graph energy to larger graphs obtained from the given by means of some graph operations.

Let $A\epsilon R^{mxn}, B\epsilon R^{pxq}$, Then the Kronecker product (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Preposition 1.1 : [6] Let $A \in M^m$ *and* $B \in M^n$. *Furthermore, let* λ *be an eigenvalues of matrix A with corresponding eigen vector x, and* μ *an eigen value of matrix B with* *corresponding eigenvector y. Then $\lambda_\mu$is an eigen value of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

2. ENERGY OF SPLITTING GRAPH

The splitting graph, $S^1(G)$ of a graph G is obtained by adding to each vertex v a new vertex v', such that $v'$is adjacent to every vertex that is adjacent to v in G. We prove the following result.

Theorem 2.1. $E\left(S^1(G)\right) = \sqrt{5}E(G)$.

$Proof$: Let $v_1, v_2 \dots \dots, v_n$be the vertices of the graph G. Then its adjacency matrix is given by

		v_1	v_2	v_3	...	v_n
	v_1	0	a_{12}	a_{13}	...	a_{1n}
	v_2	a_{21}	0	a_{23}	...	a_{2n}
$A(G)$ =	v_3	a_{31}	a_{32}	0	...	a_{3n}
	⋮	⋮	⋮	⋮	⋱	
	v_n	a_{n1}	a_{n2}	a_{n3}	...	0

Let $u_1, u_2 \dots \dots, u_n$ be the vertices corresponding to $v_1, v_2 \dots \dots, v_n$ which are added in G to obtain $S'(G)$, such that, $N(v_i) = N(u_i), i = 1,2, \dots, n$. Then $A\left(S^1(G)\right)$ can be written as a block matrix as follows

		v_1	v_2	v_3	...	v_n	u_1	u_2	u_3	...	u_n
	v_1	0	a_{12}	a_{13}	...	a_{1n}	0	a_{12}	a_{13}	...	a_{1n}
	v_2	a_{21}	0	a_{23}	...	a_{2n}	a_{21}	0	a_{23}	...	a_{2n}
	v_3	a_{31}	a_{32}	0	...	a_{3n}	a_{31}	a_{32}	0	...	a_{3n}
	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮
$A(S'(G))$	v_n	a_{1n}	a_{2n}	a_{3n}	...	0	a_{1n}	a_{2n}	a_{3n}	...	0
	u_1	0	a_{12}	a_{13}	...	a_{1n}	0	0	0	...	0
	u_2	a_{21}	0	a_{23}	...	a_{2n}	0	0	0	...	0
	u_3	a_{31}	a_{32}	0	...	a_{3n}	0	0	0	...	0
	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮
	u_n	a_{1n}	a_{2n}	a_{3n}	...		0	0	0	...	0

That is

$$A\ (S^1(G)) = \begin{bmatrix} A(G) & A(G) \\ A(G) & O \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(G)$$

Hence,

$$spec(S^1(G)) = \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)\lambda_i & \left(\frac{1-\sqrt{5}}{2}\right)\lambda_i \\ n & n \end{pmatrix}$$

Where $\lambda_i, i = 1,2, \ldots, n,$ are the eigen values of G, while $\left(\frac{1 \pm \sqrt{5}}{2}\right)$ are the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.Here,

$$E(S^1(G)) = \sum_{i=1}^{n} \left| \left(\frac{1 \pm \sqrt{5}}{2}\right)\lambda_i \right| = \sum_{i=1}^{n} |\lambda_i| \left[\frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2}\right]$$

$$= \sqrt{5} \sum_{i=1}^{n} |\lambda_i| = \sqrt{5}E(G).$$

Illustration 2.2 : Consider the cycle C_4 and its spitting graph $S^1(C_4)$.It is obvious that $E(C'_4) = 4\ as\ spec(C_4) = \begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

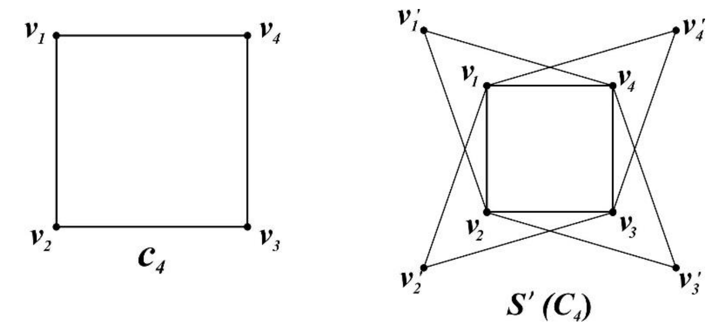


Figure 1

		v_1	v_2	v_3	v_4	v'_1	v'_2	v'_3	v'_4
	v_1	0	1	0	1	0	1	0	1
	v_2	1	0	1	0	1	0	1	0
	v_3	0	1	0	1	0	1	0	1
$A(S'(C_4)) =$	v_4	1	0	1	0	1	0	1	0
	v'_1	0	1	0	1	0	1	0	1
	v'_2	1	0	1	0	1	0	1	0
	v'_3	0	1	0	1	0	1	0	1
	v'_4	1	0	1	0	1	0	1	0

Therefore, spec $(S'(C_4) = \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} & -1-\sqrt{5} & -1+\sqrt{5} & 0 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$

Hence, $E(S'(C_4)) = 4\sqrt{5} = \sqrt{5}\,E(C_4)$

3. ENERGY OF SHADOW GRAPH

The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G' and G''. Join each vertex u' in G to the neighbours of the corresponding vertex u' in G''.

Theorem 3.1 $E(D_2(G)) = 2E(G)$

Proof : Let $v_1\ v_2\ \dots\dots\ v_n$ be the vertices of the Graph G. Then its adjacency matrix is same as in the proof Theorem 2.1. Consider the second copy of graph G with vertices v_1, v_2, $v_3\ \dots\dots u_n$ to obtain $D_2\ (G)$, such that $N(v_i), = N(u_i),\ \ i = 1,2,\dots\dots n$. Then A($D_2(G)$) can be written as a block matrix as follows :

		v_1	v_2	v_3	…	v_n	u_1	u_2	u_3	…	u_n
	v_1	0	a_{12}	a_{13}	…	a_{1n}	0	a_{12}	a_{13}	…	a_{1n}
	v_2	a_{21}	0	a_{23}	…	a_{2n}	a_{21}	0	a_{23}	…	a_{2n}
	v_3	a_{31}	a_{32}	0	…	a_{3n}	a_{31}	a_{32}	0	…	a_{3n}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$A(D_2(G)) =$	v_n	a_{1n}	a_{2n}	a_{3n}	…	0	a_{1n}	a_{2n}	a_{3n}	…	0
	u_1	0	a_{12}	a_{13}	…	a_{1n}	0	0	0	…	0
	u_2	a_{21}	0	a_{23}	…	a_{2n}	0	0	0	…	0
	u_3	a_{31}	a_{32}	0	…	a_{3n}	0	0	0	…	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	u_n	a_{1n}	a_{2n}	a_{3n}	…		0	0	0	…	0

That is,

$$A(D_2(G)) = \begin{bmatrix} A(G) & A(G) \\ A(G) & A(G) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes A(G)$$

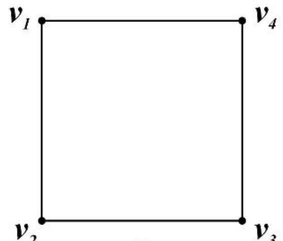
Hence,

spec$((D_2(G)) = \begin{pmatrix} 0 & 2\lambda i \\ n & n \end{pmatrix}$ where $\lambda i,\ i = 1,2\dots\dots n,$ are the eigenvalues of G, while 0, 2 are eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

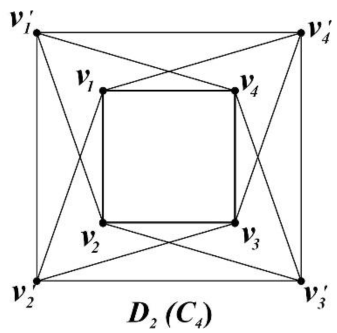
Here,

$$E\big(D_2(G)\big) = \sum_{i=1}^{n} |2\lambda_i| = 2\sum_{i=1}^{n} |\lambda_i| = 2E(G)$$

Illustration 3.2. : Consider the cycle $C_4$and its shadow graph $D_2(C_4)$. From the previous example if its known that $E(C_4) = 4$ and spec(C_4)=$\begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$



C_4



$D_2\ (C_4)$

Figure-2

		v_1	v_2	v_3	v_4	v_1'	v_2'	v_3'	v_4'
	v_1	0	1	0	1	0	1	0	1
	v_2	1	0	1	0	1	0	1	0
	v_3	0	1	0	1	0	1	0	1
$A(D_2(C_4)) =$	v_4	1	0	1	0	1	0	1	0
	v_1'	0	1	0	1	0	1	0	1
	v_2'	1	0	1	0	1	0	1	0
	v_3'	0	1	0	1	0	1	0	1
	v_4'	1	0	1	0	1	0	1	0

Therefore, $\operatorname{spec}\big(D_2(C_4)\big) = \begin{pmatrix} 4 & -4 & 0 \\ 1 & 1 & 6 \end{pmatrix}$. Hence,

$$E\big(D_2(C_4)\big) = 8 = 2E(C_4).$$

4. CONCLUDING REMARKS

The energy a graph is one of the emerging concepts within graph theory. This concept serve as a frontier between chemistry and mathematics. The energy of many graphs is known. But we have take up the problem to investigate the energy of a graph obtained by means of some graph operations on a given graph and it has been revealed that two energy of the new graph is a multiple of energy of a given graph.

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